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An analytical study of the optimum dimensions of rectangular fins and cylindrical pin fins

RONG-HUA YEH

Department of Marine Engineering and Technology, National Taiwan Ocean University, 2 Pei Ning Road, Keelung, Taiwan, Republic of China

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Abstract—Considering temperature dependent heat transfer coefficients and heat transfer from the fin tip, the optimum dimensions of rectangular fins and cylindrical pin fins are investigated analytically. In this work, the fin volume is fixed to obtain the aspect ratios of the uniform area cross-section fins with maximum heat transfer rates. The characteristic length that has been determined empirically is taken into consideration in heat transfer coefficient. The analysis shows that an optimum aspect ratio of a fin is not found for a fin with heat transfer from the tip at a large fin volume or a large heat transfer coefficient at the fin base. However, there always exists an optimum aspect ratio for an insulated-tip fin. The optimum aspect ratio of a fin is highest for a fin with an insulated tip and decreases with increasing rate of heat transfer from the tip. © 1997 Elsevier Science Ltd.

INTRODUCTION

Technology has led to a demand for high-performance, light-weight, and compact heat transfer components. To accommodate the demand, finned surfaces are used to increase the heat transfer rate between a primary surface and the surrounding fluid in heat exchange devices. Thus, optimization of the design of fins is of significant importance. For convective as well as radiative fins, Kern and Kraus [1] presented a thorough study of the optimum design of finned surfaces. A survey article by Aziz [2] provides the optimum dimensions of straight fins, annular fins, and spines of different profiles with several numerical examples included. In addition a comprehensive literature search regarding studies on extended surfaces over six decades is available in the work of Kraus [3].

In boiling heat transfer, several papers [4–7] are devoted to finding the shape of fins that would minimize the volume for a given amount of heat dissipation. However, all of these studies consider negligible heat transfer from the tip, because their fins have sharp tips. Apparently, the sharp-tip fin design has the disadvantage that the resulting profiles are too difficult to manufacture and are too fragile at the end. Several other studies [8–10] are conducted to find the dimensions of a constant thickness straight fin that would maximize the heat dissipation for a given volume. These studies, however, invoked the assumption of insulated tips.

The effect of tip convection and optimum dimensions of cooling fins is presented by Laor and Kalman [11, 12]. Using a two-dimensional analysis, Chung and Iyer [13] approximately determine the optimum aspect ratios of longitudinal rectangular fins and cylindrical pin fins. Numerous design charts are presented in their study.

The purpose of this study is to analytically determine the dimensions of uniform cross-section cylindrical-pin and longitudinal-rectangular fins with fixed fin volumes for maximum heat transfer. The effect of heat transfer from the fin tip is investigated. Also, the characteristic length is considered and is incorporated into the temperature dependent heat transfer coefficient. In addition, the validity criterion of the present one-dimensional model is proposed.

ANALYSIS

A cylindrical fin as well as a rectangular fin is now considered. The surface heat flux along the fin length varies as a power of the temperature difference between the fin and the ambient fluid. The heat transfer coefficient thus has the form:

$$h = \frac{a(T-T_a)^{m-1}}{L_c^n} \tag{1}$$

where L_c is the characteristic length and equals D or l [10, 13, and 14] for a cylindrical pin fin or a rectangular fin, respectively. In addition, m and n are dimensionless constants, while a is a dimensional constant. The values of a and m depend on the properties of the boiling liquids and the types of heat transfer [15–18]. The nature of the flow (such as laminar or turbulent flow) or the orientation of the fin is characterized by n. In addition, the following simplified assumptions have been made: (1) one-dimensional steady-state heat conduction through the fin; (2) the thermal properties of the fin are constants; (3) no heat sources or sinks exist within the fin; (4) the ambient fluid

NOMENCLATURE

A	fin profile area [m^2]
A_c	cross-section area of a fin [m^2]
b	fin thickness of a rectangular fin [m]
a	dimensional constant related to a selected heat transfer mode and location on fin surface, W m$^{-2+n}\cdot$K^{-m}
B_a	dimensionless parameter $a(T_b-T_a)^{m-1}A^{(1-n)/2}/k$, $[=\alpha^{(n+1)/2}(h_b b)/k]$
B_v	dimensionless parameter, $a(T_b-T_a)^{m-1}V^{(1-n)/3}/k$, $[=(\pi\alpha/4)^{(1-n)/3}(h_b D)/k]$
D	fin diameter of a cylindrical pin fin [m]
F	hypergeometric function
h	heat transfer coefficient, defined in equation (1) [W m$^{-2}\cdot$K^{-1}]
k	thermal conductivity (W m$^{-1}\cdot$K^{-1})
L_c	characteristic length [m]
l	fin length [m]
m	power–law exponent of temperature superheat
N	fin parameter, $\sqrt{2B_a}\alpha^{(3-n)/4}$ for a rectangular fin or $2\sqrt{B_v}[(\pi/4)^{(n-1)}\alpha^{(n+5)}]^{1/6}$ for a cylindrical pin fin
n	power–law exponent of characteristic length
Q	dimensionless heat transfer rate from a fin, $q/[(T_b-T_a)kW]$ for a rectangular fin and $a(T_b-T_a)^{m-2}qV^{-n/3}/(4\pi k^2)$ for a cylindrical fin
q	heat transfer rate from a cylindrical fin, W, or from a rectangular fin per unit width [W m^{-1}]
T	temperature [K]
V	volume of fin [m^3]
W	width of a rectangular fin [m]
X	dimensionless coordinate, x/l.

Greek symbols

α	aspect ratio of a fin, l/b for a rectangular fin or l/D for a cylindrical fin
ε	ratio of the coefficient a in heat transfer coefficient at fin tip to peripheral fin surface, a_e/a
θ	dimensionless temperature, $(T-T_a)/(T_b-T_a)$
ψ	parameter, $\sqrt{B_a(m+1)}\alpha^{(3-n)/4}$ for a rectangular fin or $(\pi/4)^{(n-1)/6}\sqrt{2B_v(m+1)}\alpha^{(n+5)/6}$ for a cylindrical fin
ξ	fin effectiveness, defined in equations (5) and (22).

Subscripts and superscripts

a	ambient or saturated
b	fin base
e	fin tip
ha	Harper–Brown approximation
max	maximum
o	optimum
*	dimensionless quantity or optimum.

temperature is uniform; (5) a constant root temperature is prescribed; and (6) the heat transfer coefficient does not vary with position from the root to the tip of the fin except insofar as h depends upon local temperature difference.

Longitudinal rectangular fins

This fin problem was investigated analytically by Liaw and Yeh [18]. Following them, the exact solution of temperature distribution in the fin is obtained as:

$$NX = \left(\frac{2\beta^{1-m}}{m+1}\right)^{1/2}\left\{\left[1-\left(\frac{\beta}{\theta}\right)^{m+1}\right]^{1/2}F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\left(\frac{\beta}{\theta}\right)^{m+1}\right]-\gamma^{1/2}F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};\gamma\right]\right\} \quad (2)$$

where

$$\beta = \theta_e(1-\gamma)^{1/(m+1)} \quad (3a)$$

and

$$\gamma = \frac{\varepsilon^2(m+1)}{4}\cdot B_a\cdot\alpha^{(-1-n)/2}\cdot\theta_e^{m-1}. \quad (3b)$$

The dimensionless heat transfer rate from the fin is derived to be:

$$Q = 2\left[\left(\frac{B_a}{m+1}\right)(1-\beta^{m+1})\right]^{1/2}\alpha^{(-1-n)/4}. \quad (4)$$

Fin effectiveness, ξ, is defined as the ratio of the heat transfer rate of a fin to that of the unfinned wall operating under the same conditions. It is derived as:

$$\xi = 2\alpha^{(1+n)/4}[B_a(m+1)]^{-1/2}(1-\beta^{m+1})^{1/2}. \quad (5)$$

For the fixed fin profile area A of a rectangular fin, it is desired to maximize Q by varying α. However, Q is a function of α and θ_e. The equivalent problem of optimizing Q will be to find the extreme values of Q subject to the constraint below:

$$H(\alpha,\theta_e) = N-\left(\frac{2\beta^{1-m}}{m+1}\right)^{1/2}\left\{(1-\beta^{m+1})^{1/2}\times F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\beta^{m+1}\right]-\gamma^{1/2}F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};\gamma\right]\right\}=0. \quad (6)$$

Note that equation (6) is obtained from equation (2) imposing the boundary condition of uniform base temperature, i.e. $\theta(1) = 1$. The solution to this problem, obtained by means of Lagrange's multiplier method, is:

$$(1-\beta_o^{m+1})^{3/2}F\left[1,\frac{m}{m+1};\frac{3}{2};1-\beta_o^{m+1}\right] - (\gamma_o\theta_{eo}^{1-m})^{1/2}(1-\beta_o^{m+1})F\left[1,\frac{m}{m+1};\frac{3}{2};\gamma_o\right] + \frac{m+1}{m-1}\cdot\left[(1-\beta_o^{m+1})^{1/2} - \frac{3-n}{1+n}\sqrt{B_a(m+1)}\cdot\alpha_o^{(3-n)/4}\beta_o^{m+1}\right] + \frac{m+1}{m+1-2m\gamma_o}(\gamma_o\theta_{eo}^{1-m})^{1/2}\left[1+\frac{2\beta_o^{m+1}}{(m-1)}\right]=0. \quad (7)$$

For given m, n, and B_a, the two optimum variables, α_o and θ_{eo}, in equation (7) are solved in the following way. Initially, θ_{eo} is guessed and α_o is obtained from equation (6). The guessed θ_{eo} and the calculated α_o are then substituted into equation (7). This procedure was continued until equation (7) is satisfied to a tolerance value of 10^{-7}.

Constant heat transfer coefficient (m = 1)

In the case of constant heat transfer coefficients, equation (7) reduces to:

$$(1-\beta_o^2)^{1/2}-\frac{3-n}{1+n}\cdot N_o(1-\gamma_o)\theta_{eo}^2+\gamma_o^{1/2}\theta_{eo}^2=0. \quad (8)$$

With the aid of the formula [19],

$$F\left[1,\frac{1}{2};\frac{3}{2};z^2\right]=\frac{1}{2z}\cdot ln\frac{1+z}{1-z},$$

equation (8) is rewritten in the form:

$$N_o = ln\frac{(1-\gamma_o^{1/2})\{1+[1-\theta_{eo}^2(1-\gamma_o)]^{1/2}\}}{\theta_{eo}(1-\gamma_o)}. \quad (9)$$

Hence, θ_{eo} is obtained as:

$$\theta_{eo} = 1/(\cosh N_o+\gamma_o^{1/2}\sinh N_o). \quad (10)$$

The substitution of equation (10) into equation (9) gives:

$$2\gamma_o^{1/2}+(1+\gamma_o)\tanh N_o-\frac{3-n}{1+n}\cdot N_o(1-\gamma_o)\sec h^2 N_o=0. \quad (11)$$

Insulated fin tip ($\varepsilon = 0$)

In this situation, it is apparent that $\gamma_o = 0$ and $\beta_o = \theta_{eo}$. Thus, equation (7) simplifies to:

$$(m-1)(1-\theta_{eo}^{m+1})^{3/2}F\left[1,\frac{m}{m+1};\frac{3}{2};1-\theta_{eo}^{m+1}\right]+(m+1)\left[(1-\theta_{eo}^{m+1})^{1/2}-\frac{3-n}{1+n}\cdot\psi_o\theta_{eo}^{m+1}\right]=0 \quad (12)$$

where $\psi_o = \sqrt{B_a(m+1)}\alpha_o^{(3-n)/4}$. Substituting equation (6), with $\varepsilon = 0$, into equation (12) yields:

$$(1+n)(m+1)+[(1+n)(m-1)-2\theta_{eo}^{m+1}(n+2m+1)]\times F\left[1,\frac{m}{m+1};\frac{3}{2};1-\theta_{eo}^{m+1}\right]=0. \quad (13)$$

It is interesting to notice that equation (13) is only a function of tip temperature for known m and n. For any given values of m and n, θ_{eo} can thus be immediately obtained.

$\varepsilon = 0$ *and* n = 0

In some applications n is zero for boiling heat transfer [15–18]. In this case, equation (6) becomes

$$\psi_o=(1-\theta_e^{m+1})^{1/2}\cdot F\left[1,\frac{m}{m+1};\frac{3}{2};1-\theta_e^{m+1}\right]. \quad (14)$$

Because the tip temperature of the optimum fin is evaluated from equation (13), ψ_o is obtained directly from equation (14), for any given m. The dimensionless optimum thickness, length, aspect ratio, and heat loss by a rectangular fin are then obtained from the following expressions:

$$b^*=\frac{b_o}{\left[A^2\left(\frac{h_b}{k}\right)\right]^{1/3}}=\left(\frac{m+1}{2\psi_o^2}\right)^{1/3} \quad (15)$$

$$l^*=\frac{l_o}{\left(\frac{kA}{h_b}\right)^{1/3}}=\left(\frac{2\psi_o^2}{m+1}\right)^{1/3} \quad (16)$$

$$\alpha^*=\alpha_o\left(\frac{h_bA^{1/2}}{k}\right)^{2/3}=\left(\frac{\psi_o^2}{m+1}\right)^{2/3} \quad (17)$$

and

$$Q^*=\frac{q_o}{(T_b-T_a)(h_b^2kA)^{1/3}}=2\left[\frac{1}{\psi_o(m+1)}\right]^{1/3}(1-\theta_e^{m+1})^{1/2}. \quad (18)$$

Harper–Brown approximation

It is interesting to explore the simplified design by using the insulated tip results with the fin length increased by one-half thickness. Because this approximation represents the heat flow rate for a rectangular fin, it should also give the dimensions of fins with maximum heat transfer rate. Following the mathematical procedures described previously, equations (4) and (6), with $\varepsilon = 0$ and the replacement of α with $\alpha + 1/2$, are coupled to solve the dimensions of fins with maximum heat transfer. The optimum aspect ratio of a fin is obtained as:

$$\alpha_o = C^* B_a^{-2/3} \tag{19}$$

where C^* is equal to 1.0773, 1.0047, 0.9392, 0.9199, 0.6529, and 0.5637 for $m = 0.75$, 1, 1.25, 1.33, 3, and 4, respectively.

Cylindrical pin fins

Following Liaw and Yeh [18], the exact expression for the one-dimensional temperature distribution in a cylindrical pin fin may be obtained. The dimensionless heat transfer rate at the base of the fin is written as:

$$Q = \frac{1}{8}\left(\frac{2B_v^3}{m+1}\right)^{1/2}\left(\frac{\pi\alpha}{4}\right)^{(n-3)/6}[1-\theta_e^{m+1}(1-\gamma)]^{1/2} \tag{20}$$

where

$$\gamma = \frac{\varepsilon^2(m+1)}{8}B_v\left(\frac{\pi\alpha}{4}\right)^{(n-1)/3}\theta_e^{m-1}. \tag{21}$$

The fin effectiveness of a cylindrical pin fin is:

$$\xi = 2\sqrt{2}[B_v(m+1)]^{-1/2}\left(\frac{\pi\alpha}{4}\right)^{(1-n)/6} \times [1-\theta_e^{m+1}(1-\gamma)]^{1/2}. \tag{22}$$

The subsidiary condition is found to be the same as equation (6) except that γ is given in equation (21) instead of equation (3b). Similar to the previous rectangular fin problem, the optimum α and θ_e are obtained by means of Lagrange's multiplier method. The resulting expression is:

$$(1-\beta_o^{m+1})^{3/2}F\left[1,\frac{m}{m+1};\frac{3}{2};1-\beta_o^{m+1}\right] - (\gamma_o\theta_{eo}^{1-m})^{1/2}(1-\beta_o^{m+1}) \times F\left[1,\frac{m}{m+1};\frac{3}{2};\gamma_o\right] + \frac{m+1}{m-1} \times\left[(1-\beta_o^{m+1})^{1/2} + \frac{n+5}{n-3}\left(\frac{\pi}{4}\right)^{(n-1)/6} \times\sqrt{2B_v(m+1)}\cdot\alpha_o^{(n+5)/6}\beta_o^{m+1}\right] + \frac{m+1}{m+1-2m\gamma_o}(\gamma_o\theta_{eo}^{1-m})^{1/2}\left[(1-\beta_o^{m+1}) - \frac{(n-1)(m+1)}{(n-3)(m-1)}\beta_o^{m+1}\right] = 0. \tag{23}$$

The solution procedure of α_o and θ_{eo} is identical to the case of a rectangular fin.

Constant heat transfer coefficient (m = 1)

In the case of the heat transfer coefficient being independent of temperature, a simplified expression may be obtained. For $m = 1$, equation (23) reduces to:

$$(n-3)(1-\beta_o^2)^{1/2} + (n+5)N_o(1-\gamma_o)\theta_{eo}^2 + (n-1)\gamma_o^{1/2}\theta_{eo}^2 = 0. \tag{24}$$

Employing the same formulas and procedures as that of the previous rectangular fin case, a relationship is found to be:

$$(n-3)[2\gamma_o^{1/2} + (1+\gamma_o)\tanh N_o] + [(n+5)N_o(1-\gamma_o) + 2\gamma_o^{1/2}]\,\mathrm{sech}^2 N_o = 0. \tag{25}$$

Insulated fin tip ($\varepsilon = 0$)

In this situation, $\gamma_o = 0$ and $\beta_o = \theta_{eo}$ in view of equations (21) and (3a). Thus, equations (6) and (23) are simplified to:

$$\psi_o = [\theta_{eo}^{1-m}(1-\theta_{eo}^{m+1})]^{1/2}F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\theta_{eo}^{m+1}\right] \tag{26}$$

and

$$(n-3)(m-1)(1-\theta_{eo}^{m+1})^{3/2} \times F\left[1,\frac{m}{m+1};\frac{3}{2};1-\theta_{eo}^{m+1}\right] + (m+1)[(n-3)(1-\theta_{eo}^{m+1})^{1/2} + (n+5)\psi_o\theta_{eo}^{m+1}] = 0 \tag{27}$$

where $\psi_o = (\pi/4)^{(n-1)/6}\sqrt{2B_v(m+1)}\alpha_o^{(n+5)/6}$. Employing the formula [19],

$$F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};z\right] = (1-z)^{(m-1)/[2(m+1)]}F\left[1,\frac{m}{m+1};\frac{3}{2};z\right]$$

and substituting equation (26) into equation (27) yields:

$$(n-3)(m+1)+[(n-3)(m-1)+2\theta_{eo}^{m+1}(n+4m+1)] \times F\left[1,\frac{m}{m+1};\frac{3}{2};1-\theta_{eo}^{m+1}\right]=0 \quad (28)$$

which is solved for θ_{eo} given m and n.

$\varepsilon = 0$ *and* $n = 0$

For further simplification of the case with $\varepsilon = 0$, the effect of n may be neglected by setting $n = 0$. This condition would be the case for boiling heat transfer. Since θ_{eo} is calculated from equation (28), ψ_o can be directly obtained from equation (26). The dimensionless optimum diameter, length, aspect ratio, and heat loss, derived from equation (26) and the definition of ψ_o, are then:

$$D^* = \frac{D_o}{\left(\frac{h_b V^2}{k}\right)^{1/5}} = 2\left(\frac{m+1}{\pi^2\psi_o^2}\right)^{1/5} \quad (29)$$

$$l^* = \frac{l_o}{\left(\frac{k^2 V}{h_b^2}\right)^{1/5}} = \left[\frac{\psi_o^4}{\pi(m+1)^2}\right]^{1/5} \quad (30)$$

$$\alpha^* = \alpha_o\left(\frac{h_b V^{1/3}}{k}\right)^{3/5} = \frac{1}{2}\left[\frac{\pi\psi_o^6}{(m+1)^3}\right]^{1/5} \quad (31)$$

and

$$Q^* = \frac{q_o}{(T_b - T_a)(h_b^4 k V^3)^{1/5}} = 2\left[\frac{\pi^2}{(m+1)\psi_o^3}\right]^{1/5}(1-\theta_{eo}^{m+1})^{1/2}. \quad (32)$$

RESULTS AND DISCUSSION

Longitudinal rectangular fins

For forced-convective external flow, n is generally 0.5 for laminar flow and 0.2 for turbulent flow [13]. The dependence of α_0 on B_a for $n = 0$, 0.2, and 0.5 and $\varepsilon = 0$, 0.5, and 1 is given in Fig. 1. This figure shows that α_o decreases with B_a for all values of n. In addition, α_o reduces drastically for a slight increase of B_a for $B_a < 0.1$ and α_o varies little for a larger B_a. At a given B_a, α_o is larger for a smaller n. Thus, the results, with the simplifying assumption of neglecting characteristic length, over-estimate α_0. For the case of a fin with heat transferred from the tip ($\varepsilon \neq 0$), the dependence of α_0 on B_a for $n = 0$, 0.2 and 0.5 is displayed in the middle and lower parts of Fig. 1. Note that two critical aspect ratios of the fins are found and a maximum B_a, designated $(B_a)_{max}$, is observed for $\varepsilon > 0$. There exists no optimum dimension ratio of a fin for $B_a > (B_a)_{max}$. This phenomenon is also observed in the two-dimensional analysis of this problem for $m = 1$ [20].

To improve our understanding, Figs. 2 and 3 present the variation of α on Q for $\varepsilon = 0$ and 1 for $m = 1$

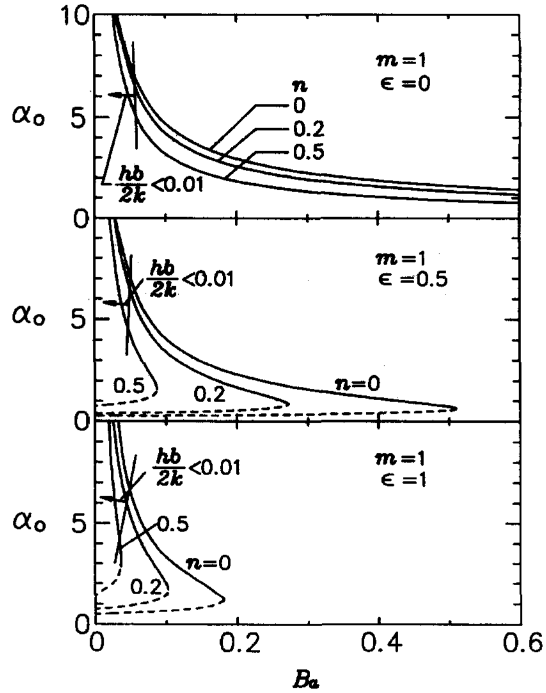

Fig. 1. Dependence of α_o on B_a for $m = 1$ and $\varepsilon = 0$, 0.5 and 1 (rectangular fin).

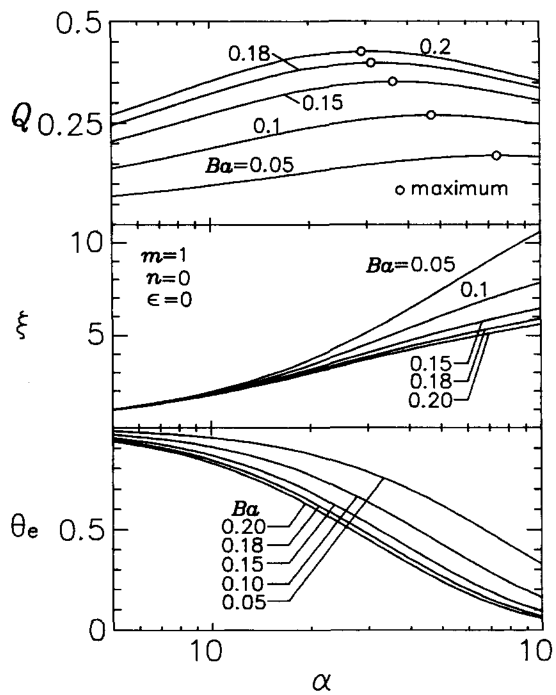

Fig. 2. The effects of B_a on heat transfer rate, fin effectiveness, and tip temperature of a rectangular fin for $m = 1$, $\varepsilon = 0$, and $n = 0$.

and $n = 0$. Inspection of Fig. 2 reveals that at a fixed B_a, Q first increases to a maximum value then decreases with increasing α. This may be apprehended that the total heat transfer area of the fin surface also increases with α for a fixed fin profile area. It is worth

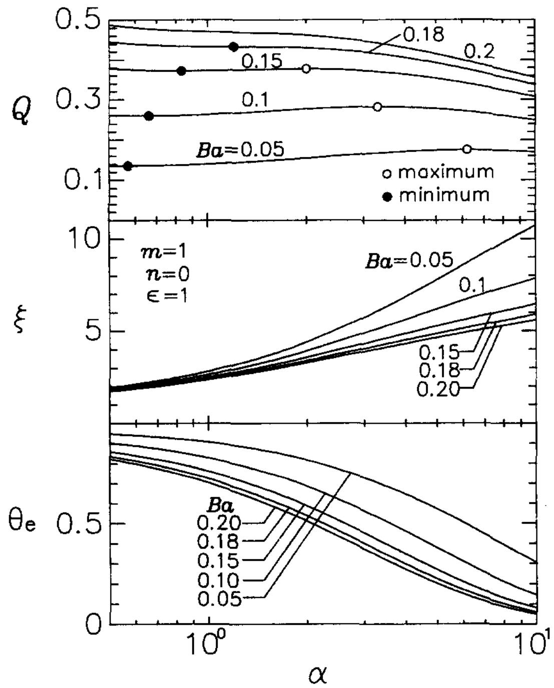

Fig. 3. The effects of B_a on heat transfer rate, fin effectiveness, and tip temperature of a rectangular fin for $m = 1$ $\varepsilon = 1$, and $n = 0$.

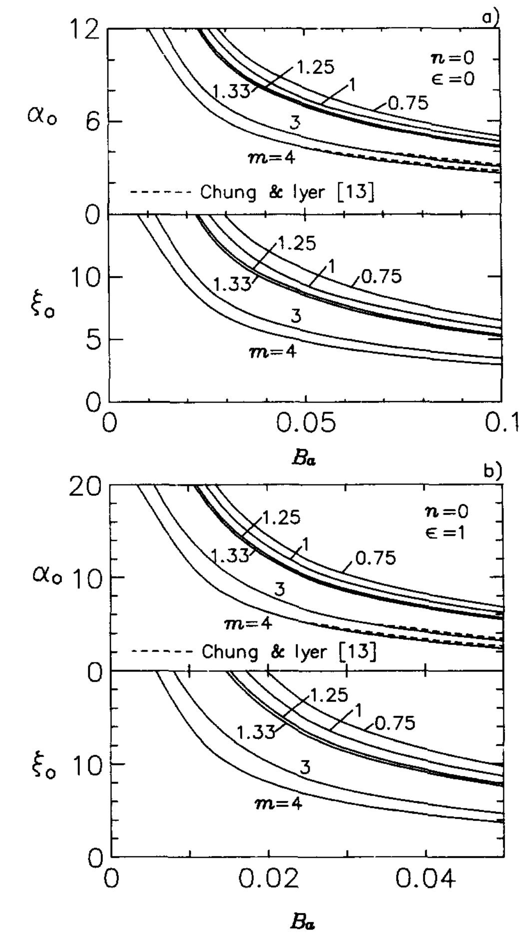

Fig. 4. Dependence of α_o and ξ_o on B_a for $m = 0.75$, 1, 1.25, 1.33, 3, and 4 (rectangular fin) for $n = 0$: (a) $\varepsilon = 0$ and (b) $\varepsilon = 1$.

noting that θ_e decreases with increasing α. Hence, with the conflicting trends of increasing heat transfer area and decreasing the temperature difference between fin surface and environment, a maximum heat transfer rate may occur at a certain α which is designated α_o in this study. For a smaller B_a, the temperature drop along the fin is smaller on increasing α. Hence, the aspect ratio of a fin with maximum heat transfer becomes larger. For a fin with heat transferred from the tip, the decrease in tip temperature is very slight whereas the effective heat transfer area decreases on increasing α in the regions of smaller α, as seen in Fig. 3. A minimum heat transfer rate of a fin thus may occur. This is different from the case of a fin with an insulated tip because the minimum heat dissipation is apparently at a smaller α because of no heat transfer from the fin tip for a fixed fin profile area. To further increase α, a significant increase in the heat transfer area on fin surface but large temperature drop along the fin is observed. Hence, the heat transfer rate from a fin reaches a maximum then decreases with increasing α. In the Fig. 1, it is then understood that the larger α_0 (indicated in solid lines) of a fin yields the maximum heat transfer whereas the smaller α_o (indicated in dashed lines) of a fin gives a minimum heat dissipation. In this study, only the aspect ratios of fins with maximum heat transfer rates are of interest.

At $B_a > 0.18$ $[=(B_a)_{max}]$, the heat dissipation of a fin is very large at a smaller α due to the fact that a large heat transfer coefficient exists at the fin base. In addition, the temperature drop along the fin is larger for a larger B_a. Although the surface area of a fin increases with α, the heat dissipation of a fin decreases with α because the reduction in the temperature difference between the fin surface and the ambient fluid is much larger on increasing α. Hence, no optimum aspect ratio of a fin exists for $B_a \geqslant (B_a)_{max}$. Also, the effects of B_a on fin effectiveness are also included in Figs. 2 and 3. Comparison of the two figures shows that a slight difference in ξ at a larger α whereas a pronounced difference in ξ at a smaller α between $\varepsilon = 0$ and 1 are observed. This is because the heat transfer at the fin tip is large for a short and stubby fin.

Aside from the constant heat transfer coefficient case ($m = 1$), other important heat transfer modes are film boiling ($m = 0.75$), laminar free convection ($m = 1.25$), turbulent free convection ($m = 1.33$), nucleate boiling ($m = 3$), and radiation into free space at zero temperature ($m = 4$). The aspect ratio and effectiveness of an optimum rectangular fin for $\varepsilon = 0$ and 1 are depicted in Fig. 4. Note that the optimum aspect ratio of a fin decreases with increasing m at fixed B_a. This does not mean the optimum radiative fin is shorter and fatter than the optimum convective fin because the coefficient of a in the temperature

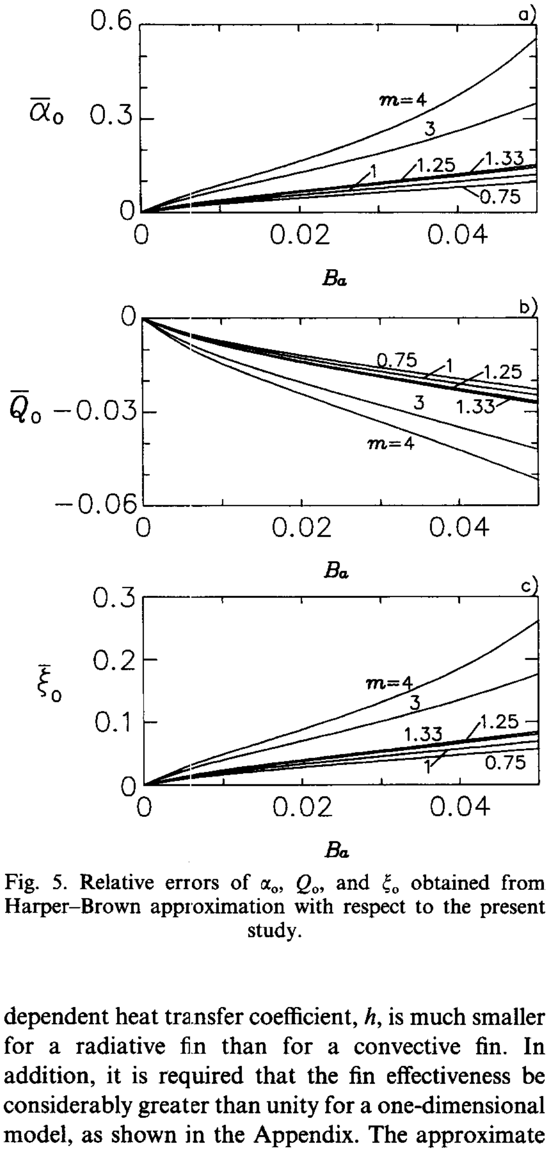

Fig. 5. Relative errors of α_o, Q_o, and ξ_o obtained from Harper–Brown approximation with respect to the present study.

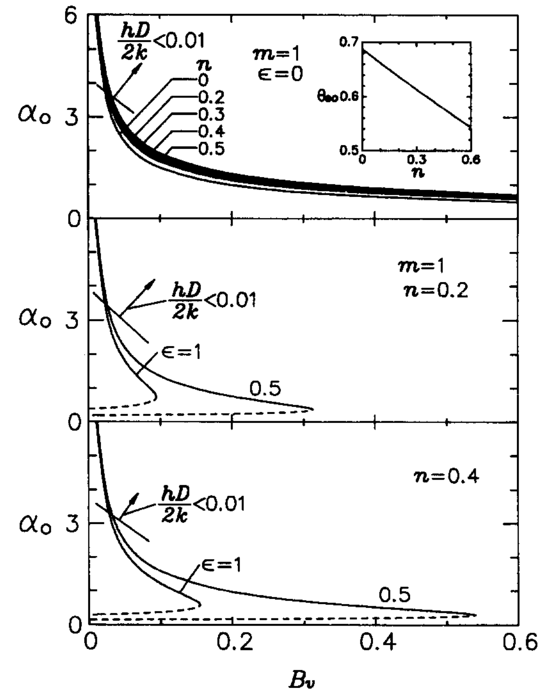

Fig. 6. Dependence of α_o on B_v for $m = 1$ and $\varepsilon = 0$, 0.5, and 1 (cylindrical pin fin).

dependent heat transfer coefficient, h, is much smaller for a radiative fin than for a convective fin. In addition, it is required that the fin effectiveness be considerably greater than unity for a one-dimensional model, as shown in the Appendix. The approximate two-dimensional solutions of α_o calculated by Chung and Iyer [13] are also given in this figure. It is observed that the predicted values from Chung and Iyer completely overlap those of this study for $m = 0.75$, 1, 1.25, 1.33.

Figure 5 shows the relative errors of the results obtained from the Harper–Brown approximation with respect to the present analytical study. Note that $\bar{\alpha}_o$, $\bar{Q}_o$, and $\bar{\xi}_o$ denote $(\alpha_{o,ha}-\alpha_o)/\alpha_o$, $(Q_{o,ha}-Q_o)/Q_o$, and $(\xi_{o,ha}-\xi_o)/\xi_o$, respectively. It is observed that the heat loss by a fin predicted from Harper–Brown approximation is lower than the exact value and the percent error is less than 5.5% up to $B_a = 0.05$; however, the error rates of α_o and ξ_o are large for a larger B_a especially for a larger m. This is due to the fact that the fins with maximum heat transfer are found to be long and slender at a smaller B_a.

Cylindrical pin fin

The heat transfer coefficient is independent of fin surface temperature but is inversely proportional to D^n for $m = 1$. In forced convection, n is generally 0.5 for laminar flow and 0.2 or 0.3 for turbulent flow. Figure 6 shows the dependence of α_o on B_v for $m = 1$ and for $\varepsilon = 0$, 0.5, and 1. Similar to the case of a rectangular fin, the optimum aspect ratio of a cylindrical fin decreases with increasing B_v for all values of n. A maximum B_v exists for $\varepsilon > 0$ and no solution is found for a fin with B_v greater than $(B_v)_{max}$.

For the boiling heat transfer case ($n = 0$), the dependence of α_o and ξ_o on B_v for $\varepsilon = 0$ and 1 is displayed as Fig. 7. Analogous to the rectangular fin case, both α_o and ξ_o decrease with increasing B_v. For a given heat transfer mode, α_o decreases with increasing ε at a fixed B_v. Also, to see the differences between this model and a two-dimensional one, the data predicted by Chung and Iyer [13] is presented in this figure.

To examine the influence of ε on α_o, the optimum aspect ratios of cylindrical fins for $n = 0$ and $B_v = 0.01$ are shown in Fig. 8. As can be seen, α_o decreases with ε whereas Q_o increases with ε. This is observed for all heat transfer modes.

The effect of n and m on $(B_a)_{max}$ and $(B_v)_{max}$ are given in Fig. 9(a) and (b). From Fig. 9(a), it is seen that $(B_a)_{max}$ decreases drastically with increasing ε for smaller ε. For $m = 1$, $(B_a)_{max}$ increases with decreasing n at a fixed ε. For $n = 0$, $(B_a)_{max}$ increases with decreasing m at a fixed ε. As ε tends to zero, $(B_a)_{max}$ approaches infinity. Thus, the optimum rectangular fin only exists at a smaller B_a for a larger ε. There always exists an optimum aspect ratio of a fin with an insulated tip which has been clarified in the first paragraph of this section. Also, note that large errors

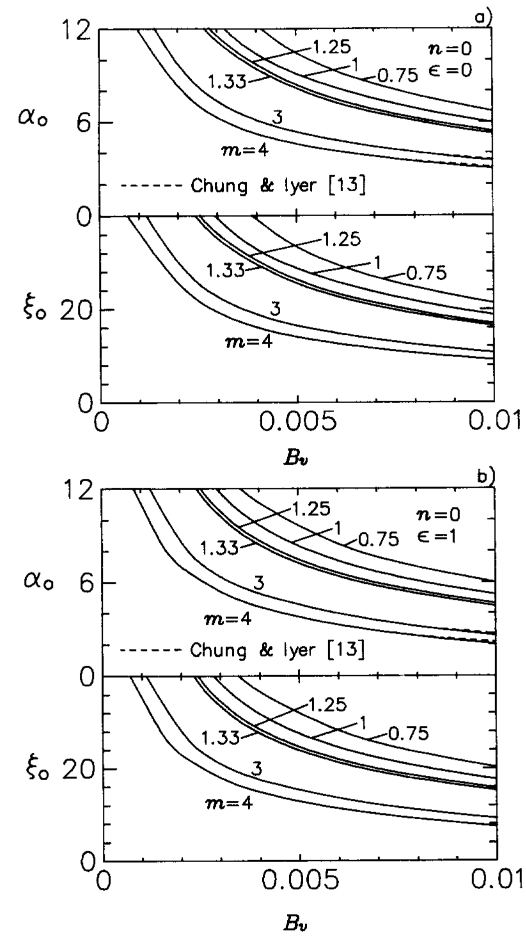

Fig. 7. Dependence of α_o and ξ_o on B_v for $m = 0.75$, 1, 1.25, 1.33, 3, and 4 (cylindrical pin fin) with the case of $n = 0$: (a) $\varepsilon = 0$ and (b) $\varepsilon = 1$.

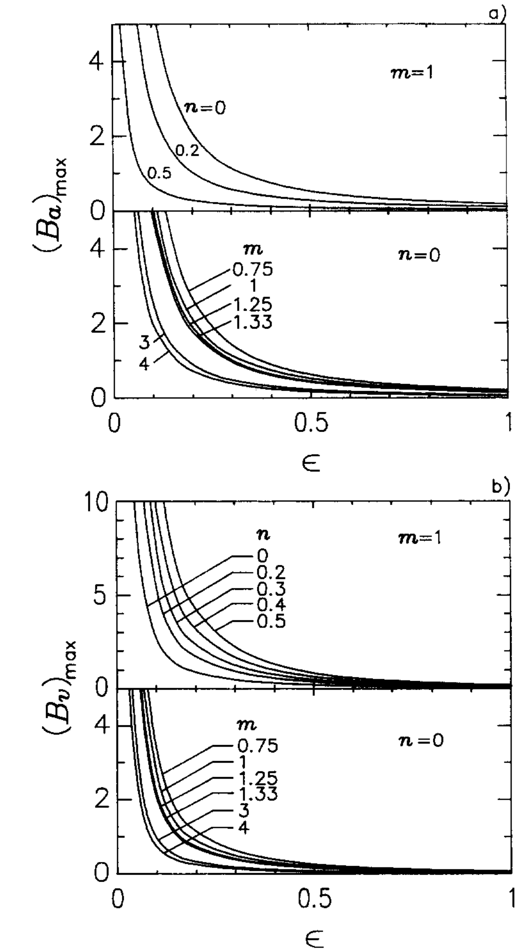

Fig. 9. Dependence of (a) $(B_a)_{max}$ and (b) $(B_v)_{max}$ on ε for various m and n.

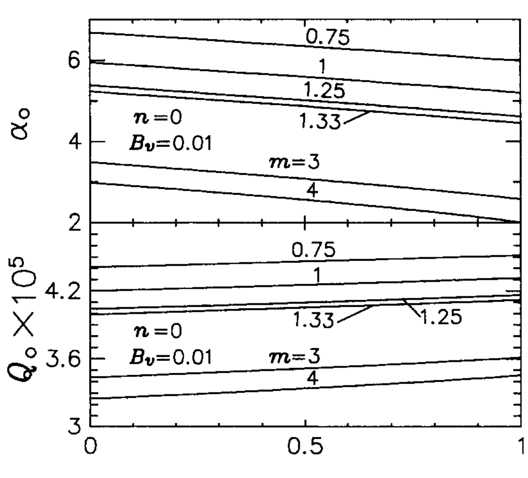

Fig. 8. Dependence of α_o and Q_o on ε for $B_v = 0.01$ and $n = 0$ (cylindrical pin fin).

may occur for α_0 calculated at a B_a or B_v near $(B_a)_{max}$ or $(B_v)_{max}$ [20]. Hence, it is suggested that B_a or B_v should not be close to $(B_a)_{max}$ or $(B_v)_{max}$ for an accurate design. In addition, note that for $m = 1$ $(B_v)_{max}$ increases with n at a fixed ε in Fig. 9(b). This phenomenon shows an inverse effect to the case of a rectangular fin, i.e. $(B_a)_{max}$ increases with decreasing n.

CONCLUSIONS

(1) In forced convection, the tip temperatures of optimum fins with insulated tips depend on n only. For $n = 0$ and $\varepsilon = 0$, the tip temperatures of optimum fins are functions of m only.

(2) There always exists an optimum aspect ratio of a fin with an insulated tip. The aspect ratio of an optimum fin decreases with increasing fin volume or heat transfer coefficient at the fin base.

(3) For a fin with heat transferred from the tip, the increase in aspect ratio will first reduce the total heat transfer area but the temperature difference between the fin surface and the ambient fluid varies little for smaller aspect ratios of fins. Upon increasing the fin's aspect ratio, the surface area of a fin increases very quickly just as the temperature drop along the fin does. Hence, first a minimum and then a maximum heat dissipation of a fin is obtained on increasing the fin's aspect ratio. At a larger fin volume or heat trans-

fer coefficient at fin base, especially $B_a > (B_a)_{max}$ or $B_v > (B_v)_{max}$, the temperature drop along the fin is significant. The increase in aspect ratio of a fin will not improve fin's heat transfer. Thus, no optimum aspect ratio of a fin is found under this condition.

(4) At a specified heat transfer mode, the optimum aspect ratio of a fin is the largest for a fin with an insulated tip and decreases with increasing heat transfer rate from the tip.

(5) In forced as well as free convection, the optimum aspect ratio of a cylindrical fin increases with n at a given B_v whereas α_o decreases with n at a fixed B_a for a rectangular fin.

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APPENDIX

The one-dimensional conduction model is valid only when the longitudinal heat flux is far greater than the transversal heat flux. This requirement may be written as:

$$\frac{q}{A_c} \gg h_b(T_b - T_a). \tag{A1}$$

Note that equation (A1) can also be expressed as:

$$\xi \gg 1. \tag{A2}$$

Substituting equation (5) into the above criterion gives:

$$2\alpha^{(1+n)/4}\left[\frac{1-\beta^{m+1}}{B_a(m+1)}\right]^{1/2} \gg 1 \tag{A3}$$

for a rectangular fin. After some rearrangement, equation (A3) becomes:

$$1 \gg \frac{B_a(m+1)}{4}\cdot\alpha^{-(1+n)/2} + \beta^{m+1}. \tag{A4}$$

The above criterion can be simply expressed as:

$$1 \gg \frac{h_b b(m+1)}{4k} \tag{A5}$$

since β is always positive. For $m = 1$, it is easy to obtain the familiar criterion, i.e. $1 \gg h_b b/(2k)$.